

# Iterated function systems, representations, and Hilbert space

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## Abstract

In this paper, we are concerned with spectral-theoretic features of general iterated function systems (IFS). Such systems arise from the study of iteration limits of a finite family of maps  $\tau_i$ ,  $i = 1, \dots, N$ , in some Hausdorff space  $Y$ . There is a standard construction which generally allows us to reduce to the case of a compact invariant subset  $X \subset Y$ . Typically, some kind of contractivity property for the maps  $\tau_i$  is assumed, but our present considerations relax this restriction. This means that there is then not a natural equilibrium measure  $\mu$  available which allows us to pass the point-maps  $\tau_i$  to operators on the Hilbert space  $L^2(\mu)$ . Instead, we show that it is possible to realize the maps  $\tau_i$  quite generally in Hilbert spaces  $\mathcal{H}(X)$  of square-densities on  $X$ . The elements in  $\mathcal{H}(X)$  are equivalence classes of pairs  $(\varphi, \mu)$ , where  $\varphi$  is a Borel function on  $X$ ,  $\mu$  is a positive Borel measure on  $X$ , and  $\int_X |\varphi|^2 d\mu < \infty$ . We say that  $(\varphi, \mu) \sim (\psi, \nu)$  if there is a positive Borel measure  $\lambda$  such that  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ , and

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}}, \quad \lambda \text{ a.e. on } X.$$

We prove that, under general conditions on the system  $(X, \tau_i)$ , there are isometries

$$S_i: (\varphi, \mu) \longmapsto (\varphi \circ \sigma, \mu \circ \tau_i^{-1})$$

in  $\mathcal{H}(X)$  satisfying  $\sum_{i=1}^N S_i S_i^* = I =$  the identity operator in  $\mathcal{H}(X)$ . For the construction we assume that some mapping  $\sigma: X \rightarrow X$  satisfies the conditions  $\sigma \circ \tau_i = \text{id}_X$ ,  $i = 1, \dots, N$ .

We further prove that this representation in the Hilbert space  $\mathcal{H}(X)$  has several universal properties.

*Key words:* Hilbert space, Cuntz algebra, completely positive map, creation operators, wavelet packets, pyramid algorithm, product measures, orthogonality relations, equivalence of measures, iterated function systems (IFS), scaling function, multiresolution, subdivision scheme, singular measures, absolutely continuous measures

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## 1 Introduction

In this paper we are concerned with *iterated function systems* (IFS) and their representation in Hilbert space. For contractive IFS's, there is a known standard construction of a family of measures, and Hilbert spaces induced by these measures. However, the constructions are not universal in any reasonable sense, and they only admit a very restricted family of covariant measures.

Let  $X$  be a compact metric space, and let  $\tau_i: X \rightarrow X$ ,  $i = 1, \dots, N$ , satisfy

$$d(\tau_i(x), \tau_i(y)) \leq C d(x, y), \quad i = 1, \dots, N, \quad x, y \in X, \quad (1.1)$$

for some  $C$ ,  $0 < C < 1$ . Let  $p_i > 0$  be given such that  $\sum_{i=1}^N p_i = 1$ . Then it follows from a theorem of Hutchinson [1] that there is a unique positive Borel measure  $\mu = \mu_{(p)}$  on  $X$  such that  $\mu(X) = 1$ , and

$$\sum_{i=1}^N p_i \mu \circ \tau_i^{-1} = \mu, \quad (1.2)$$

where the measures  $\mu \circ \tau_i^{-1}$  are defined by  $\mu \circ \tau_i^{-1}(E) := \mu(\tau_i^{-1}(E))$ ,  $E \in \mathcal{B}(X)$  = the Borel subsets of  $X$ , where

$$\tau_i^{-1}(E) := \{x \in X \mid \tau_i(x) \in E\}. \quad (1.3)$$

We shall need a “variable-coefficient version” of (1.2) which is motivated by applications to wavelets; see [2] and [3]. In this version of (1.2), there is a whole family of measures  $\mu_f$  indexed by vectors  $f$  in some complex Hilbert space  $\mathcal{K}$ , and moreover there is a finite family of isometries  $S_i: \mathcal{K} \rightarrow \mathcal{K}$ ,  $i = 1, \dots, N$ , such that

$$\sum_{i=1}^N S_i S_i^* = I_{\mathcal{K}}, \quad (1.4)$$

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and (1.2) takes the form

$$\sum_{i=1}^N \mu_{S_i^* f} \circ \tau_i^{-1} = \mu_f. \quad (1.5)$$

Isometries  $S_i$  subject to (1.4) are said to satisfy the Cuntz relations, or to define a representation of the Cuntz algebra  $\mathcal{O}_N$ ; see [4]. The algebra  $\mathcal{O}_N$  is a simple  $C^*$ -algebra, and its representations are ubiquitous in analysis and applied mathematics. A special class of these relations is known to define sub-band filters in signal processing, to define subdivision algorithms in computer graphics, and to define effective pyramid algorithms in wavelet analysis; see [3], [5]. However, the classical approach to subdivisions via (1.2) is known not to suffice for the representation of wavelet systems, see [6], not even for the simplest quadrature mirror filters which are used for the standard Haar wavelet or for the Daubechies wavelets.

Readers not familiar with wavelets may pick up the essentials from Chapter 5 of the classic [7]. More current results, presented from the viewpoint of operator theory, may be found in Chapter 2 of the monograph [3], or in the survey paper [5].

Another aim of the present paper is to relax the contractivity condition (1.1). Our starting point is a compact Hausdorff space  $X$  and continuous maps  $\sigma: X \rightarrow X$ ,  $\tau_i: X \rightarrow X$ ,  $i = 1, \dots, N$ , such that

$$\sigma \circ \tau_i = \text{id}_X. \quad (1.6)$$

It follows from (1.6) that  $\sigma$  is onto, and that each  $\tau_i$  is one-to-one. We will be especially interested in the case when there are distinct branches  $\tau_i: X \rightarrow X$  such that

$$\bigcup_{i=1}^N \tau_i(X) = X. \quad (1.7)$$

For such systems, we show in Section 4 that there is a *universal* representation of  $\mathcal{O}_N$  in a Hilbert space  $\mathcal{H}(X)$  which is functorial, is naturally defined, and contains every representation of  $\mathcal{O}_N$ .

The elements in the universal Hilbert space  $\mathcal{H}(X)$  are equivalence classes of pairs  $(\varphi, \mu)$  where  $\varphi$  is a Borel function on  $X$  and where  $\mu$  is a positive Borel measure on  $X$ . We will set  $\varphi\sqrt{d\mu} := \text{class}(\varphi, \mu)$  for reasons which we spell out below.

While our present methods do adapt to the more general framework when the space  $X$  of (1.6)–(1.7) is not assumed compact, but only  $\sigma$ -compact, we will still restrict the discussion here to the compact case. This is for the sake of simplicity of the technical arguments. But we encourage the reader to follow our proofs below, and to formulate for him/herself the corresponding results

when  $X$  is not necessarily assumed compact. Moreover, if  $X$  is not compact, then there is a variety of special cases to take into consideration, various abstract notions of “escape to infinity”. We leave this wider discussion for a later investigation, and we only note here that our methods allow us to relax the compactness restriction on  $X$ .

There is a classical construction in operator theory which lets us realize point transformations in Hilbert space. It is called the Koopman representation; see, for example, [8, p. 135]. But this approach only applies if the existence of invariant, or quasi-invariant, measures is assumed. In general such measures are not available. The present paper proposes a different way of realizing families of point transformations in Hilbert space in a general context where no such assumptions are made. Our Hilbert spaces are motivated by a construction due to S. Kakutani [9], L. Schwartz, and E. Nelson [10], among others. The reader is also referred to an updated presentation of the measure-class Hilbert spaces due to Tsirelson [11] and Arveson [12, Chapter 14].

We say that  $(\varphi, \mu) \sim (\psi, \nu)$  if there is a third positive Borel measure  $\lambda$  on  $X$  such that  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ , and

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}}, \quad \lambda \text{ a.e. on } X, \quad (1.8)$$

where  $\ll$  denotes relative absolute continuity, and where  $d\mu/d\lambda$  denotes the usual Radon-Nikodym derivative, i.e.,  $d\mu/d\lambda \in L^1(\lambda)$ , and  $d\mu = (d\mu/d\lambda) d\lambda$ .

In Section 2, we review some basic properties of the Hilbert space  $\mathcal{H}(X)$ . This space is called the Hilbert space of  $\sigma$ -functions, or square densities, and it was studied for different reasons in earlier papers of L. Schwartz, E. Nelson [10], and W. Arveson [13].

Our first new result is the fact that the isometries  $S_i: \mathcal{H}(X) \rightarrow \mathcal{H}(X)$  are defined by

$$S_i: (\varphi, \mu) \longmapsto (\varphi \circ \sigma, \mu \circ \tau_i^{-1}), \quad (1.9)$$

or equivalently,  $S_i: \varphi \sqrt{d\mu} \mapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau_i^{-1}}$ , and that the operators satisfy the Cuntz relations (1.4).

Note that, at the outset, it is not even clear *a priori* that  $S_i$  in (1.9) defines a transformation of  $\mathcal{H}(X)$ . To verify this, we will need to show that if two equivalent pairs are substituted on the left-hand side in (1.9), then they produce equivalent pairs as output, on the right-hand side. Recalling the definition (1.8) of the equivalence relation  $\sim$ , there is no obvious or intuitive reason for why this should be so.

To stress the intrinsic transformation rules of  $\mathcal{H}(X)$ , the vectors in  $\mathcal{H}(X)$

are usually denoted  $\varphi\sqrt{d\mu}$  rather than  $(\varphi, \mu)$ . This is a suggestive notation which motivates the definition of the inner product of  $\mathcal{H}(X)$ . It is also helpful in understanding Theorem 4.2 below. If  $\varphi\sqrt{d\mu}$  and  $\psi\sqrt{d\nu}$  are in  $\mathcal{H}(X)$ , we define their Hilbert inner product by

$$\left\langle \varphi\sqrt{d\mu} \mid \psi\sqrt{d\nu} \right\rangle := \int_X \bar{\varphi} \psi \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda, \quad (1.10)$$

where  $\lambda$  is some positive Borel measure, chosen such that  $\mu \ll \lambda$  and  $\nu \ll \lambda$ . For example, we could take  $\lambda = \mu + \nu$ . To be in  $\mathcal{H}(X)$ ,  $\varphi\sqrt{d\mu}$  must satisfy

$$\left\| \varphi\sqrt{d\mu} \right\|^2 = \int_X |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda = \int_X |\varphi|^2 d\mu < \infty. \quad (1.11)$$

## 2 Isometries in $\mathcal{H}(X)$

In this preliminary section we prove three general facts about the process of inducing operators in the Hilbert space  $\mathcal{H}(X)$  from underlying point transformations in  $X$ . The starting point is a given continuous mapping  $\sigma: X \rightarrow X$ , mapping onto  $X$ . We will be concerned with the special case when  $X$  is a compact Hausdorff space, and when there is one or more continuous branches  $\tau_i: X \rightarrow X$  of the inverse, i.e., when

$$\sigma \circ \tau_i = \text{id}_X. \quad (2.1)$$

Recall that elements in  $\mathcal{H}(X)$  are equivalence classes of pairs  $(\varphi, \mu)$  where  $\varphi$  is a Borel function on  $X$ ,  $\mu$  is a positive Borel measure on  $X$ , and  $\int_X |\varphi|^2 d\mu < \infty$ . An equivalence class will be denoted  $\varphi\sqrt{d\mu}$ , and we show that there are isometries

$$S_i: \varphi\sqrt{d\mu} \longmapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau_i^{-1}}, \quad (2.2)$$

with orthogonal ranges in the Hilbert space  $\mathcal{H}(X)$ . Moreover, we calculate an explicit formula for the adjoint co-isometries  $S_i^*$ .

In the next section, we shall then restrict the setting to the special case of measures  $\mu$  such that  $\mu \circ \tau_i^{-1} \ll \mu$ , where  $\ll$  stands for “absolutely continuous with respect to”.

**Lemma 2.1** *Let  $X$  be a compact Hausdorff space, and let the mapping  $\sigma: X \rightarrow X$  be onto. Suppose  $\tau: X \rightarrow X$  satisfies  $\sigma \circ \tau = \text{id}_X$ . Assume that both  $\sigma$  and  $\tau$  are continuous. Let  $\mathcal{H} = \mathcal{H}(X)$  be the Hilbert space of classes  $(\varphi, \mu)$  where  $\varphi$  is a Borel function on  $X$  and  $\mu$  is a positive Borel measure such that  $\int |\varphi|^2 d\mu < \infty$ . The equivalence relation is defined in the usual way: two pairs  $(\varphi, \mu)$  and  $(\psi, \nu)$  are said to be equivalent, written  $(\varphi, \mu) \sim (\psi, \nu)$ , if for some*

positive measure  $\lambda$ ,  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ , we have the following identity:

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}} \quad (\text{a.e. } \lambda). \quad (2.3)$$

Then there is an isometry  $S: \mathcal{H} \rightarrow \mathcal{H}$  which is well defined by the assignment

$$S((\varphi, \mu)) := (\varphi \circ \sigma, \mu \circ \tau^{-1}), \quad (2.4)$$

or

$$S: \varphi \sqrt{d\mu} \mapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau^{-1}},$$

where  $\mu \circ \tau^{-1}(E) := \mu(\tau^{-1}(E))$ , and  $\tau^{-1}(E) := \{x \in X \mid \tau(x) \in E\}$ , for  $E \in \mathcal{B}(X)$ .

**PROOF.** We leave the verification of the following four facts to the reader; see also [10].

- (i) If  $\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}}$  for some  $\lambda$  such that  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ , and if some other measure  $\lambda'$  satisfies  $\mu \ll \lambda'$ ,  $\nu \ll \lambda'$ , then

$$\varphi \sqrt{\frac{d\mu}{d\lambda'}} = \psi \sqrt{\frac{d\nu}{d\lambda'}} \quad (\text{a.e. } \lambda').$$

- (ii) The “vectors” in  $\mathcal{H}$  are equivalence classes of pairs  $(\varphi, \mu)$  as described in the statement of the lemma. For two elements  $(\varphi, \mu)$  and  $(\psi, \nu)$  in  $\mathcal{H}$ , define the sum by

$$(\varphi, \mu) + (\psi, \nu) := \left( \phi \sqrt{\frac{d\mu}{d\lambda}} + \psi \sqrt{\frac{d\nu}{d\lambda}}, \lambda \right), \quad (2.5)$$

where  $\lambda$  is a positive Borel measure satisfying  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ . The sum in (2.5) is also written  $\varphi \sqrt{d\mu} + \psi \sqrt{d\nu}$ . The definition of the sum (2.5) passes through the equivalence relation  $\sim$ , i.e., we get an equivalent result on the right-hand side in (2.5) if equivalent pairs are used as input on the left-hand side. A similar conclusion holds for the definition (2.6) below of the inner product  $\langle \cdot \mid \cdot \rangle$  in the Hilbert space  $\mathcal{H}$ .

- (iii) Scalar multiplication,  $c \in \mathbb{C}$ , is defined by  $c(\varphi, \mu) := (c\varphi, \mu)$ , and the Hilbert space inner product is

$$\left\langle \varphi \sqrt{d\mu} \mid \psi \sqrt{d\nu} \right\rangle = \langle (\varphi, \mu) \mid (\psi, \nu) \rangle := \int_X \bar{\varphi} \psi \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda \quad (2.6)$$

where  $\mu \ll \lambda$ ,  $\nu \ll \lambda$ .

- (iv) It is known, see [10], that  $\mathcal{H}$  is a Hilbert space. In particular, it is complete: if a sequence  $(\varphi_n, \mu_n)$  in  $\mathcal{H}$  satisfies

$$\lim_{n,m \rightarrow \infty} \|(\varphi_n, \mu_n) - (\varphi_m, \mu_m)\|^2 = 0,$$

then there is a pair  $(\varphi, \mu)$  with

$$\int_X |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda = \int_X |\varphi|^2 d\mu < \infty, \quad (2.7)$$

where

$$\lambda := \sum_{n=1}^{\infty} 2^{-n} \mu_n (X)^{-1} \mu_n, \quad (2.8)$$

and  $\|(\varphi, \mu) - (\varphi_n, \mu_n)\|^2 \xrightarrow{n \rightarrow \infty} 0$ .

Assuming that the expression in (2.4) defines an operator  $S$  in  $\mathcal{H}$ , it follows from (2.5) that  $S$  is linear. To see this, let  $(\varphi, \mu)$ ,  $(\psi, \nu)$ , and  $\lambda$  be as stated in the conditions below (2.5). Then  $\mu \circ \tau^{-1} \ll \lambda \circ \tau^{-1}$ , and  $\nu \circ \tau^{-1} \ll \lambda \circ \tau^{-1}$ , and a calculation shows that the following formula holds for the transformation of the Radon-Nikodym derivatives: setting

$$\frac{d\mu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}} = k_\mu, \quad (2.9)$$

we have

$$k_\mu \circ \tau = \frac{d\mu}{d\lambda} \quad (\text{a.e. } \lambda). \quad (2.10)$$

Similarly  $k_\nu := \frac{d\nu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$  satisfies

$$k_\nu \circ \tau = \frac{d\nu}{d\lambda} \quad (\text{a.e. } \lambda). \quad (2.11)$$

To show that  $S$  is linear, we must calculate the sum

$$(\varphi \circ \sigma, \mu \circ \tau^{-1}) + (\psi \circ \sigma, \nu \circ \tau^{-1}), \quad (2.12)$$

or, in expanded notation, we must verify that

$$(\varphi \circ \sigma \sqrt{k_\mu} + \psi \circ \sigma \sqrt{k_\nu}, \lambda \circ \tau^{-1}) \sim \left( \left( \varphi \sqrt{\frac{d\mu}{d\lambda}} + \psi \sqrt{\frac{d\nu}{d\lambda}} \right) \circ \sigma, \lambda \circ \tau^{-1} \right). \quad (2.13)$$

We get this class identity by an application of (2.10) as follows:

$$k_\mu(x) = k_\mu(\tau(\sigma(x))) = \left( \sqrt{\frac{d\mu}{d\lambda}} \circ \sigma \right) \Big|_{\tau(X)}(x) \quad (\text{a.e. } \lambda \circ \tau^{-1}).$$

Similarly, for the other measure, we get

$$k_\nu = \left( \sqrt{\frac{d\nu}{d\lambda}} \circ \sigma \right) \Big|_{\tau(X)} \quad (\text{a.e. } \lambda \circ \tau^{-1}). \quad (2.14)$$

Assuming again that  $S$  in (2.4) is well defined, we now show that it is isometric, i.e., that  $\|S(\varphi, \mu)\|^2 = \|(\varphi, \mu)\|^2$ , referring to the norm of  $\mathcal{H}$ . In view of (2.5) and (2.13), it is enough to show that

$$\int_X |\varphi \circ \sigma|^2 k_\mu d\lambda \circ \tau^{-1} = \int_X |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda. \quad (2.15)$$

But, using (2.10), we get

$$\int_X |\varphi \circ \sigma|^2 k_\mu d\lambda \circ \tau^{-1} = \int_X |\varphi \circ \sigma \circ \tau|^2 k_\mu \circ \tau d\lambda \stackrel{(2.10)}{=} \int_X |\varphi|^2 \frac{d\mu}{d\lambda} d\lambda,$$

which is the desired formula (2.15).

It remains to prove that  $S$  is well defined, i.e., that the following implication holds:

$$(\varphi, \mu) \sim (\psi, \nu) \implies (\varphi \circ \sigma, \mu \circ \tau^{-1}) \sim (\psi \circ \sigma, \nu \circ \tau^{-1}). \quad (2.16)$$

To do this, we go through a sequence of implications which again uses the fundamental transformation rules (2.10) and (2.14).

Pick some  $\lambda$  such that  $\mu \ll \lambda$  and  $\nu \ll \lambda$ . We establish the following implication:

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}} \quad (\text{a.e. } \lambda) \implies (\varphi \circ \sigma) \sqrt{k_\mu} = (\psi \circ \sigma) \sqrt{k_\nu} \quad (\text{a.e. } \lambda \circ \tau^{-1}), \quad (2.17)$$

where  $k_\mu = \frac{d\mu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$  and  $k_\nu = \frac{d\nu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$ . The desired conclusion (2.16) follows from this.

We now turn to the proof of the implication (2.17). We pick a third measure  $\lambda$  as described, and assume the identity

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}} \quad \text{a.e. } \lambda.$$

Let  $f$  be a bounded Borel function on  $X$ . In the following calculations, all integrals are over the full space  $X$ , but the measures change as we make transformations, and we use the definition of the Radon-Nikodym formula.



First note that

$$\begin{aligned} \int f k_\mu \left( \frac{d\nu}{d\lambda} \circ \sigma \right) d\lambda \circ \tau^{-1} &= \int f \left( \frac{d\nu}{d\lambda} \circ \sigma \right) d\mu \circ \tau^{-1} \\ &= \int f \circ \tau \frac{d\nu}{d\lambda} d\mu = \int f \circ \tau \frac{d\nu}{d\lambda} \frac{d\mu}{d\lambda} d\lambda. \end{aligned}$$

But by symmetry, we also have

$$\int f k_\nu \left( \frac{d\mu}{d\lambda} \circ \sigma \right) d\lambda \circ \tau^{-1} = \int f \circ \tau \frac{d\nu}{d\lambda} \frac{d\mu}{d\lambda} d\lambda.$$

Putting the last two formulas together, we arrive at the following identity:

$$\int_X f k_\mu \frac{d\nu}{d\lambda} \circ \sigma d\lambda \circ \tau^{-1} = \int_X f k_\nu \frac{d\mu}{d\lambda} \circ \sigma d\lambda \circ \tau^{-1}.$$

Since the function  $f$  is arbitrary, we get

$$k_\mu \left( \frac{d\nu}{d\lambda} \circ \sigma \right) = k_\nu \left( \frac{d\mu}{d\lambda} \circ \sigma \right) \quad \text{a.e. } \lambda \circ \tau^{-1}$$

and, of course,

$$\sqrt{k_\mu} \sqrt{\frac{d\nu}{d\lambda} \circ \sigma} = \sqrt{k_\nu} \sqrt{\frac{d\mu}{d\lambda} \circ \sigma} \quad \text{a.e. } \lambda \circ \tau^{-1}.$$

Using now the identity

$$\varphi \sqrt{\frac{d\mu}{d\lambda}} = \psi \sqrt{\frac{d\nu}{d\lambda}} \quad \text{a.e. } \lambda,$$

we arrive at the formula

$$\varphi \circ \sigma \sqrt{k_\mu} \sqrt{\frac{d\mu}{d\lambda} \circ \sigma} \sqrt{\frac{d\nu}{d\lambda} \circ \sigma} = \psi \circ \sigma \sqrt{k_\nu} \sqrt{\frac{d\mu}{d\lambda} \circ \sigma} \sqrt{\frac{d\nu}{d\lambda} \circ \sigma},$$

and by cancellation,

$$\varphi \circ \sigma \sqrt{k_\mu} = \psi \circ \sigma \sqrt{k_\nu} \quad \text{a.e. } \lambda \circ \tau^{-1}.$$

This completes the proof of the implication (2.17), and therefore also of (2.16). This means that if the linear operator  $S$  is defined as in (2.4), then the result is independent of which element is chosen in the equivalence class represented by the pair  $(\varphi, \mu)$ . Putting together the steps in the proof, we conclude that  $S: \mathcal{H} \rightarrow \mathcal{H}$  is an isometry, and that it has the properties which are stated in the lemma.  $\square$

**Lemma 2.2** *Let  $X$  be a compact Hausdorff space, and let  $\sigma$  be as in the statement of Lemma 2.1, i.e.,  $\sigma: X \rightarrow X$  is onto and continuous. Suppose  $\sigma$  has two distinct branches of the inverse, i.e.,  $\tau_i: X \rightarrow X$ ,  $i = 1, 2$ , continuous, and satisfying  $\sigma \circ \tau_i = \text{id}_X$ ,  $i = 1, 2$ . Let  $S_i: \mathcal{H} \rightarrow \mathcal{H}$  be the corresponding isometries, i.e.,*

$$S_i((\varphi, \mu)) := (\varphi \circ \sigma, \mu \circ \tau_i^{-1}), \quad (2.18)$$

or

$$S_i: \varphi \sqrt{d\mu} \mapsto \varphi \circ \sigma \sqrt{d\mu \circ \tau_i^{-1}}.$$

Then the two isometries have orthogonal ranges, i.e.,

$$\langle S_1((\varphi, \mu)) \mid S_2((\psi, \nu)) \rangle = 0 \quad (2.19)$$

for all pairs of vectors in  $\mathcal{H}$ , i.e., all  $(\varphi, \mu) \in \mathcal{H}$  and  $(\psi, \nu) \in \mathcal{H}$ .

**PROOF.** Note that in the statement (2.19) of the conclusion, we use  $\langle \cdot \mid \cdot \rangle$  to denote the inner product of the Hilbert space  $\mathcal{H}$ , as it was defined in (2.6).

With the two measures  $\mu$  and  $\nu$  given, then the expression in (2.19) involves the transformed measures  $\mu \circ \tau_1^{-1}$  and  $\nu \circ \tau_2^{-1}$ . Now pick some measure  $\lambda$  such that  $\mu \circ \tau_1^{-1} \ll \lambda$  and  $\nu \circ \tau_2^{-1} \ll \lambda$ . Then the expression in (2.19) is

$$\int_X \overline{\varphi \circ \sigma} \psi \circ \sigma \sqrt{\frac{d\mu \circ \tau_1^{-1}}{d\lambda}} \sqrt{\frac{d\nu \circ \tau_2^{-1}}{d\lambda}} d\lambda. \quad (2.20)$$

But  $\frac{d\mu \circ \tau_1^{-1}}{d\lambda}$  is supported on  $\tau_1(X)$ , while  $\frac{d\nu \circ \tau_2^{-1}}{d\lambda}$  is supported on  $\tau_2(X)$ . Since  $\tau_1(X) \cap \tau_2(X) = \emptyset$  by the choice of distinct branches for the inverse of  $\sigma$ , we conclude that the integral in (2.20) vanishes.  $\square$

In the next lemma we prove a formula for the adjoint  $S^*$  of the isometry  $S$  which was introduced in Lemma 2.1. Now  $S^*$  refers to the inner product (2.6) which is given at the outset, and which defines the Hilbert space  $\mathcal{H}$ .

**Lemma 2.3** *Let  $X$ ,  $\sigma$ , and  $\tau$  be given as in the statement of Lemma 2.1, i.e., we assume that  $\sigma$  is onto, that both  $\sigma$  and  $\tau$  are continuous, and that*

$$\sigma \circ \tau = \text{id}_X. \quad (2.21)$$

Let  $S$  be the isometry defined in (2.4), and let  $S^*$  be the adjoint co-isometry. Then

$$S^*((\varphi, \mu)) = (\varphi \circ \tau, \mu \circ \sigma^{-1}) \quad (2.22)$$

for all  $(\varphi, \mu) \in S\mathcal{H}$ .

**PROOF.** Recall that operators in  $\mathcal{H}$  are defined on equivalence classes: just as in the proof of Lemma 2.1, we must check the implication

$$(\varphi, \mu) \sim (\psi, \nu) \implies (\varphi \circ \tau, \mu \circ \sigma^{-1}) \sim (\psi \circ \tau, \nu \circ \sigma^{-1}). \quad (2.23)$$

While the verification of (2.23) involves the transformation rules for Radon-Nikodym derivatives, the steps of the proof are quite analogous to the arguments from the proof of Lemma 2.1, and they are left to the reader.

Now let  $T$  denote the operator on  $\mathcal{H}$  which is defined by the formula (2.22). It is clear that  $TS = I =$  the identity operator in  $\mathcal{H}$ , i.e., that  $TS(\varphi, \mu) = (\varphi, \mu)$  for all  $(\varphi, \mu) \in \mathcal{H}$ . Indeed,

$$TS(\varphi, \mu) = T(\varphi \circ \sigma, \mu \circ \tau^{-1}) = (\varphi \circ \sigma \circ \tau, \mu \circ \tau^{-1} \circ \sigma^{-1}) = (\varphi, \mu),$$

where the identity (2.21) was used in the last step of the argument.

The assertion of the lemma is that  $T|_{S\mathcal{H}} = S^*$ . Since  $TS = I$ , and  $S$  is isometric, we need only set  $T$  equal to zero on  $(S\mathcal{H})^\perp$ , where

$$(S\mathcal{H})^\perp = \{x \in \mathcal{H} \mid \langle Sy \mid x \rangle = 0, y \in \mathcal{H}\}. \quad (2.24)$$

Let  $x = \text{class}(\psi, \nu) \in (S\mathcal{H})^\perp$ , and let  $y = \text{class}(\varphi, \mu)$ . The argument from the proof of Lemma 2.1 shows that there is a positive Borel measure  $\lambda$  such that  $\mu \circ \tau^{-1} \ll \lambda \circ \tau^{-1}$  and  $\nu \ll \lambda \circ \tau^{-1}$ . Recall that the Radon-Nikodym derivative  $k_\mu = \frac{d\mu \circ \tau^{-1}}{d\lambda \circ \tau^{-1}}$  satisfies

$$k_\mu \circ \tau = \frac{d\mu}{d\lambda}. \quad (2.25)$$

We now calculate the inner-product term from (2.24):

$$\begin{aligned} \langle Sy \mid x \rangle &= \int_X \overline{\varphi \circ \sigma} \, \psi \, \sqrt{k_\mu} \, \sqrt{\frac{d\nu}{d\lambda \circ \tau^{-1}}} \, d\lambda \circ \tau^{-1} \\ &= \int_X \bar{\varphi} \, \psi \circ \tau \, \sqrt{k_\mu \circ \tau} \, \sqrt{\frac{d\nu}{d\lambda \circ \tau^{-1}} \circ \tau} \, d\lambda \\ &= \int_X \bar{\varphi} \, \psi \circ \tau \, \sqrt{\frac{d\mu}{d\lambda}} \, \sqrt{\frac{d\nu}{d\lambda \circ \tau^{-1}} \circ \tau} \, d\lambda, \end{aligned}$$

where we used (2.25) in the last step.

Since this expression  $\equiv 0$  for all  $(\varphi, \mu) \in \mathcal{H}$ , and  $d\mu = \frac{d\mu}{d\lambda} d\lambda$ , we conclude that  $\psi \circ \tau = 0$  a.e.  $\lambda$ .  $\square$

### 3 Representations in $L^2(\mu)$

Let  $X$  be a compact Hausdorff space, and let  $\mu$  be a positive Borel measure on  $X$ . For simplicity, we assume that  $\mu$  is normalized, i.e., that  $\mu(X) = 1$ . Ideally, we look for some measure  $\mu$  such that the Hilbert space  $L^2(\mu) = L^2(X, \mu)$  suffices for the representation theory under discussion. For representations of the Cuntz algebras  $\mathcal{O}_N$ , it may be possible to stay within the Hilbert space  $L^2(\mu)$ , while for some other representations, the “larger” Hilbert space  $\mathcal{H}$  of Section 2 is forced on us.

The general setting in this section will be the same as in Section 2: the transformations  $\sigma, \tau: X \rightarrow X$  are assumed continuous, and  $\tau$  is a branch of the inverse of  $\sigma$ , i.e., we assume that

$$\sigma \circ \tau = \text{id}_X. \quad (3.1)$$

It follows that  $\sigma$  is onto, and that  $\tau$  is one-to-one. We will show in this section that if

$$\mu \circ \tau^{-1} \ll \mu, \quad (3.2)$$

then the isometry  $S$  from Lemma 2.1 may be realized in  $L^2(\mu)$ .

**Theorem 3.1** *Let  $\sigma, \tau: X \rightarrow X$  be continuous, and suppose that  $\sigma \circ \tau = \text{id}_X$  holds. Let  $\mu$  be a positive Borel measure on  $X$  such that  $\mu(X) = 1$  and*

$$\mu \circ \tau^{-1} \ll \mu. \quad (3.3)$$

*Let  $S: \mathcal{H} \rightarrow \mathcal{H}$  be the isometry defined in Lemma 2.1, i.e.,*

$$S((\varphi, \mu)) := (\varphi \circ \sigma, \mu \circ \tau^{-1}). \quad (3.4)$$

*Setting*

$$S_\mu \varphi := \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau^{-1}}{d\mu}} \quad (3.5)$$

*and*

$$W_\mu \varphi := (\varphi, \mu), \quad (3.6)$$

*we get two isometries,  $S_\mu: L^2(\mu) \rightarrow L^2(\mu)$  and  $W_\mu: L^2(\mu) \rightarrow \mathcal{H}$ , such that*

$$W_\mu S_\mu = S W_\mu. \quad (3.7)$$

**PROOF.** The assertion (3.7) states that  $W_\mu$  intertwines the two isometries  $S_\mu$  and  $S$ , or, expressed as a diagram, that the commutativity shown in Fig.

$$\begin{array}{ccc}
L^2(\mu) & \xrightarrow{W_\mu} & \mathcal{H} \\
S_\mu \downarrow & & \downarrow S \\
L^2(\mu) & \xrightarrow{W_\mu} & \mathcal{H}
\end{array}$$

Fig. 1.  $W_\mu S_\mu = S W_\mu$ .

1 holds. To prove that  $S_\mu$  is isometric, note that

$$\begin{aligned}
\int_X |S_\mu \varphi| d\mu &= \int_X |\varphi \circ \sigma|^2 \frac{d\mu \circ \tau^{-1}}{d\mu} d\mu = \int_X |\varphi \circ \sigma|^2 d\mu \circ \tau^{-1} \\
&= \int_X |\varphi \circ \sigma \circ \tau|^2 d\mu = \int_X |\varphi|^2 d\mu.
\end{aligned}$$

It is clear from the definition of the norm in  $\mathcal{H}$  that  $W_\mu$  is isometric. To verify (3.7), we note that

$$W_\mu S_\mu \varphi = \left( \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau^{-1}}{d\mu}}, \mu \right),$$

and that

$$S W_\mu \varphi = (\varphi \circ \sigma, \mu \circ \tau^{-1}).$$

But since  $\mu \circ \tau^{-1} \ll \mu$ , it is clear from (2.3) that

$$\left( \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau^{-1}}{d\mu}}, \mu \right) \sim (\varphi \circ \sigma, \mu \circ \tau^{-1}).$$

Since the vectors in  $\mathcal{H}$  are equivalence classes, the desired intertwining identity (3.7) holds.  $\square$

**Corollary 3.2** *Let  $X$  be a compact Hausdorff space, and let  $\sigma, \tau: X \rightarrow X$  satisfy the conditions stated in Theorem 3.1. Let  $\mu$  be a positive Borel measure on  $X$  such that  $\mu(X) = 1$  and  $\mu \circ \tau^{-1} \ll \mu$ . Then the Radon-Nikodym derivative*

$$p_\mu := \frac{d\mu \circ \tau^{-1}}{d\mu} \tag{3.8}$$

*satisfies*

$$p_\mu \geq 1 \quad \mu\text{-a.e. on } \tau(X), \tag{3.9}$$

$$S_\mu^* \varphi = \varphi \circ \tau (p_\mu \circ \tau)^{-1/2}. \tag{3.10}$$

**PROOF.** Since  $S_\mu: L^2(\mu) \rightarrow L^2(\mu)$  is isometric by the theorem,  $S_\mu^*$  is contractive in  $L^2(\mu)$ , and  $\|S_\mu^*\| = 1$ . But a substitution of formula (3.10) yields

$\langle \varphi | S_\mu \psi \rangle = \langle S_\mu^* \varphi | \psi \rangle$  for  $\varphi, \psi \in L^2(\mu)$ . Indeed, we have the following identity:

$$\begin{aligned} \int \bar{\varphi} \psi \circ \sigma p_\mu^{1/2} d\mu &= \int \bar{\varphi} \psi \circ \sigma p_\mu^{-1/2} p_\mu d\mu \\ &= \int \bar{\varphi} \psi \circ \sigma p_\mu^{-1/2} d\mu \circ \tau^{-1} = \int \overline{\varphi \circ \tau} \psi (p_\mu \circ \tau)^{-1/2} d\mu. \end{aligned}$$

This proves formula (3.10) for the co-isometry  $S_\mu^*: L^2(\mu) \rightarrow L^2(\mu)$ .  $\square$

**Corollary 3.3** *Let  $X$  be a compact Hausdorff space, and let  $N \in \mathbb{N}$ ,  $N \geq 2$ , be given. Let  $\sigma: X \rightarrow X$  be continuous and onto. Suppose there are  $N$  distinct branches of the inverse, i.e., continuous  $\tau_i: X \rightarrow X$ ,  $i = 1, \dots, N$ , such that*

$$\sigma \circ \tau_i = \text{id}_X. \quad (3.11)$$

*Suppose there is a positive Borel measure  $\mu$  such that  $\mu(X) = 1$ , and*

$$\mu \circ \tau_i^{-1} \ll \mu, \quad i = 1, \dots, N. \quad (3.12)$$

*Then the isometries*

$$S_i \varphi := \varphi \circ \sigma \sqrt{\frac{d\mu \circ \tau_i^{-1}}{d\mu}} \quad (3.13)$$

*satisfy*

$$\sum_{i=1}^N S_i S_i^* = I_{L^2(\mu)} \quad (3.14)$$

*if and only if*

$$\bigcup_{i=1}^N \tau_i(X) = X. \quad (3.15)$$

**PROOF.** We already know from Lemma 2.2 that the isometries  $S_i: L^2(\mu) \rightarrow L^2(\mu)$  are mutually orthogonal, i.e., that

$$S_i^* S_j = \delta_{i,j} I_{L^2(\mu)}. \quad (3.16)$$

It follows that the terms in the sum (3.14) are commuting projections. Hence

$$\sum_{i=1}^N S_i S_i^* \leq I_{L^2(\mu)}. \quad (3.17)$$

Moreover, we conclude that (3.14) holds if and only if

$$\sum_{i=1}^N \|S_i^* \varphi\|^2 = \|\varphi\|^2, \quad \varphi \in L^2(\mu). \quad (3.18)$$

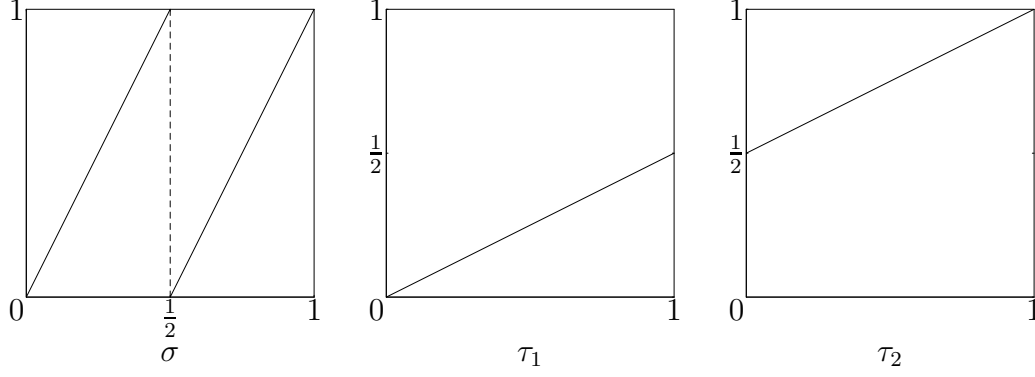


Fig. 2. Subdivisions of the unit interval.

Setting  $p_i := \frac{d\mu \circ \tau_i^{-1}}{d\mu}$ , we get

$$S_i^* \varphi = \varphi \circ \tau_i (p_i \circ \tau_i)^{-1/2}; \quad (3.19)$$

see (3.10) of the previous corollary. We get

$$\|S_i^* \varphi\|^2 = \int_X |\varphi \circ \tau_i|^2 (p_i \circ \tau_i)^{-1} d\mu = \int_{\tau_i(X)} |\varphi|^2 p_i^{-1} d\mu \circ \tau_i^{-1} = \int_{\tau_i(X)} |\varphi|^2 d\mu.$$

Recall that the branches  $\tau_i$  of the inverse are distinct. So in view of (3.10), the sets  $\tau_i(X)$  are non-overlapping. The equivalence (3.14)  $\Leftrightarrow$  (3.15) now follows directly from the previous calculation.  $\square$

**Example 3.4** Let  $X$  be the unit interval  $[0, 1)$  and let  $\mu$  be the restricted Lebesgue measure. Let  $\sigma(x) = 2x \bmod 1$ , and set  $\tau_1(x) = \frac{x}{2}$ ,  $\tau_2(x) = \frac{x+1}{2}$ . The graphs of the three maps are illustrated in Figure 2. Then the two isometries

$$S_1 \varphi(x) = \varphi(\sigma(x)) \sqrt{2} \chi_{[0, 1/2)}(x), \quad S_2 \varphi(x) = \varphi(\sigma(x)) \sqrt{2} \chi_{[1/2, 1)}(x), \quad (3.20)$$

and their adjoints

$$S_1^* \varphi(x) = \frac{1}{\sqrt{2}} \varphi\left(\frac{x}{2}\right), \quad S_2^* \varphi(x) = \frac{1}{\sqrt{2}} \varphi\left(\frac{x+1}{2}\right), \quad (3.21)$$

satisfy the relations (3.14) and (3.16), i.e., they define a representation of the Cuntz algebra  $\mathcal{O}_2$  on the Hilbert space  $L^2(0, 1)$ . Note that the conditions (3.12) are satisfied since

$$\mu \circ \tau_1^{-1} = 2\mu|_{[0, 1/2)} \quad \text{and} \quad \mu \circ \tau_2^{-1} = 2\mu|_{[1/2, 1)}. \quad (3.22)$$

**Example 3.5** Let  $X$  be the middle-third Cantor set. The mapping  $\sigma(x) = 3x \bmod 1$  restricts to  $X$  when  $X$  is embedded in the unit interval in the usual fashion; see Fig. 3. The two maps  $\tau_1(x) = \frac{x}{3}$ ,  $\tau_2(x) = \frac{x+2}{3}$  satisfy  $X = \tau_1(X) \cup \tau_2(X)$ , and

$$\sigma \circ \tau_i = \text{id}_X. \quad (3.23)$$

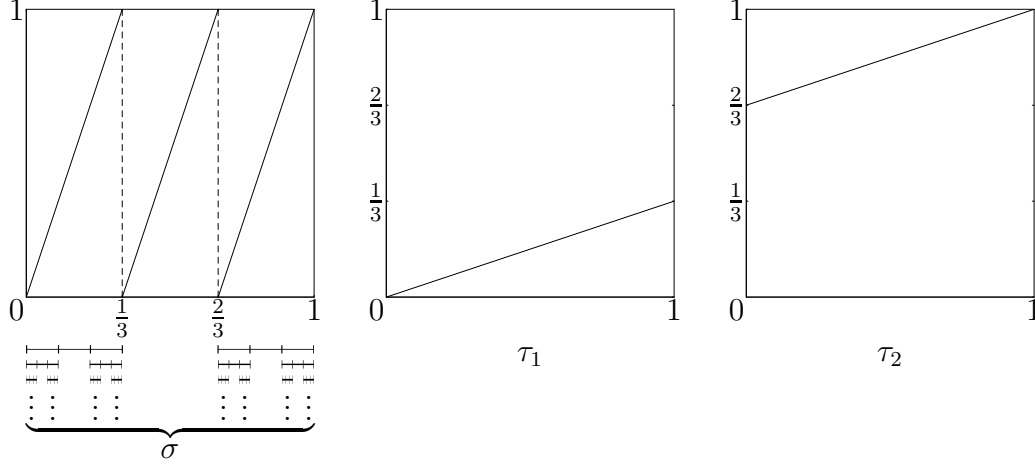


Fig. 3. The Cantor set  $X$ .

The geometry of  $X$  is illustrated in Fig. 3. The Cantor measure  $\mu$  is determined uniquely by the two properties

$$\mu(X) = 1 \quad \text{and} \quad \mu = \frac{1}{2} (\mu \circ \tau_1^{-1} + \mu \circ \tau_2^{-1}). \quad (3.24)$$

In fact, we have

$$\mu \circ \tau_i^{-1} = 2\mu|_{\tau_i(X)}, \quad (3.25)$$

and (3.12) are clearly satisfied. The difference between the two examples is that now  $\mu$  is the Cantor measure, while in the previous example it was the Lebesgue measure. The support of the Cantor measure is the Cantor set  $X$ .

Using the corollary, we note that two isometries  $S_i: L^2(\mu) \rightarrow L^2(\mu)$  are defined by the formulas

$$S_i \varphi(x) = \varphi(\sigma(x)) \sqrt{2} \chi_{\tau_i(X)}(x), \quad (3.26)$$

where now  $\sigma(x) = 3x \bmod 1$ , and

$$\tau_1(X) = X \cap \left[0, \frac{1}{3}\right] \quad \text{and} \quad \tau_2(X) = X \cap \left[\frac{2}{3}, 1\right].$$

The formulae for the adjoint co-isometries  $S_i^*: L^2(\mu) \rightarrow L^2(\mu)$  are

$$S_1^* \varphi(x) = \frac{1}{\sqrt{2}} \varphi\left(\frac{x}{3}\right), \quad S_2^* \varphi(x) = \frac{1}{\sqrt{2}} \varphi\left(\frac{x+2}{3}\right), \quad (3.27)$$

and it is immediate that the Cuntz relations (3.14) and (3.16) are satisfied: by direct verification, or by an application of Corollary 3.3, we note that the isometries (3.26) form a representation of  $\mathcal{O}_2$  on the Hilbert space  $L^2(\mu)$ .



## 4 From representations to iterated function systems

In Section 2 we showed that every iterated function system (IFS), even if not contractive, may be represented in the Hilbert space  $\mathcal{H}$  of equivalence classes  $\text{class}(\varphi, \mu)$ , where  $\varphi$  is a Borel function and  $\mu$  is a positive Borel measure. The equivalence relation  $\sim$  which defines  $\mathcal{H}$  is given by (2.3). In Section 3 we specialized this construction to the case when measures  $\mu$  may be found such that

$$\mu \circ \tau_i^{-1} \ll \mu, \quad (4.1)$$

where  $\tau_i$ ,  $i = 1, \dots, N$ , is the given IFS. In that case, we proved that the resulting representation of  $\mathcal{O}_N$  may be realized in  $L^2(\mu)$ . For each such measure  $\mu$  satisfying (4.1), the representation of  $\mathcal{O}_N$  on  $L^2(\mu)$  is a sub-representation of the “global” representation on the Hilbert space  $\mathcal{H}$  from Section 2.

In this section, the tables are turned: now the starting point is some representation of  $\mathcal{O}_N$ , and we wish to reconstruct some IFS and its realization in Hilbert space.

**Definition 4.1** Let  $N \in \mathbb{N}$ ,  $N \geq 2$ , and let  $\mathcal{O}_N$  be the Cuntz algebra on  $N$  generators, i.e., the  $C^*$ -algebra based on the relations (3.14) and (3.16). It is known, see [4], to be a simple  $C^*$ -algebra. If  $\mathcal{K}$  is a Hilbert space, we say that  $\mathcal{O}_N$  is represented on  $\mathcal{K}$  if there are isometries  $S_i: \mathcal{K} \rightarrow \mathcal{K}$  such that

$$S_i^* S_j = \delta_{i,j} I_{\mathcal{K}} \quad \text{and} \quad \sum_{i=1}^N S_i S_i^* = I_{\mathcal{K}}. \quad (4.2)$$

When  $\mathcal{K}$  is given at the outset, we will denote all the representations of  $\mathcal{O}_N$  on  $\mathcal{K}$  by  $\text{Rep}(\mathcal{O}_N, \mathcal{K})$ .

If  $X$  is a compact Hausdorff space, then we denote by  $\mathcal{H}(X)$  the Hilbert space of equivalence classes  $\text{class}(\varphi, \mu)$  introduced in Section 2. If further  $\sigma, \tau_i: X \rightarrow X$  is a given IFS of continuous maps satisfying the three conditions

$$\sigma \circ \tau_i = \text{id}_X, \quad (4.3)$$

$$\text{the maps } \tau_i \text{ are distinct branches of the inverse for } \sigma: X \rightarrow X, \text{ and} \quad (4.4)$$

$$\bigcup_{i=1}^N \tau_i(X) = X, \quad (4.5)$$

then the isometries

$$S_i: (\varphi, \mu) \longrightarrow (\varphi \circ \sigma, \mu \circ \tau_i^{-1}), \quad (4.6)$$

or

$$S_i \left( \varphi \sqrt{d\mu} \right) := \varphi \circ \sigma \sqrt{d\mu \circ \tau_i^{-1}},$$

define an element in  $\text{Rep}(\mathcal{O}_N, \mathcal{H}(X))$ , and we say that this is the *universal* representation of  $\mathcal{O}_N$  built on the IFS  $(X, \tau_i)$ .

Our next result justifies this terminology.

If  $\mathcal{K}$  and  $\mathcal{H}$  are Hilbert spaces which both carry representations of  $\mathcal{O}_N$ , we say that the representation on  $\mathcal{K}$  is a *sub-representation* of that on  $\mathcal{H}$  if there is an isometry  $W: \mathcal{K} \rightarrow \mathcal{H}$  such that

$$W S_i^{\mathcal{K}} = S_i^{\mathcal{H}} W. \quad (4.7)$$

The isometry  $W$  is said to *intertwine* the two representations. (The superscripts in the formula (4.7) indicate the Hilbert space on which the isometries  $S_i$  act.)

**Theorem 4.2** *Let  $N \in \mathbb{N}$ ,  $N \geq 2$ , be given. Let  $(S_i)$  be in  $\text{Rep}(\mathcal{O}_N, \mathcal{K})$  for some Hilbert space  $\mathcal{K}$ , and let  $(X, \sigma, \tau_i)$  be an iterated function system on a compact metric space  $X$  which satisfies the conditions of Definition 4.1, and furthermore*

$$\text{diameter}(\tau_{i_1} \circ \tau_{i_2} \circ \cdots \circ \tau_{i_k}(X)) \xrightarrow[k \rightarrow \infty]{} 0. \quad (4.8)$$

*Then  $(S_i, \mathcal{K})$  is a sub-representation of the universal representation of  $\mathcal{O}_N$  on  $\mathcal{H}(X)$ , i.e., there is an isometry  $W: \mathcal{K} \rightarrow \mathcal{H}(X)$  such that*

$$W S_i^{\mathcal{K}} = S_i W, \quad (4.9)$$

*where the isometries  $S_i: \mathcal{H}(X) \rightarrow \mathcal{H}(X)$  are defined as in (4.6).*

**PROOF.** Let  $N, \mathcal{K}, X, \sigma, \tau_i$  be given as in the statement of the theorem, and let  $(S_i) \in \text{Rep}(\mathcal{O}_N, \mathcal{K})$ . We shall omit the superscript when referring to the isometries  $S_i (= S_i^{\mathcal{K}})$ . It is easy to see that for every  $k \in \mathbb{N}$ , the  $N^k$  distinct projections

$$P(i_1, \dots, i_k) := S_{i_1} S_{i_2} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_2}^* S_{i_1}^* \quad (4.10)$$

are mutually orthogonal, and that

$$\sum_{i_1, \dots, i_k} P(i_1, \dots, i_k) = I_{\mathcal{K}} \quad (4.11)$$

and

$$\sum_{j=1}^N P(i_1, \dots, i_k, j) = P(i_1, \dots, i_k). \quad (4.12)$$

Using an argument from [2], it follows that there is a unique projection-valued measure  $P(\cdot)$  defined on the Borel sets in  $X$  such that

$$P(\tau_{i_1} \circ \tau_{i_2} \circ \cdots \circ \tau_{i_k}(X)) = P(i_1, \dots, i_k), \quad (4.13)$$

where the notation is abused:  $P$  is denoting both the expression in (4.10) and the new measure. Specifically,  $P(\cdot)$  satisfies the following additional five properties:

- (i)  $I_{\mathcal{K}} = P(X) = \int_X P(dx)$ ;
- (ii)  $P(E) = P(E)^* = P(E)^2$ ,  $E \in \mathcal{B}(X)$ ;
- (iii)  $P(\cdot)$  is countably additive;
- (iv)  $P(E)P(F) = 0$  if  $E, F \in \mathcal{B}(X)$  and  $E \cap F = \emptyset$ ;
- (v)  $\sum_{i=1}^N S_i P(\tau_i^{-1}(E)) S_i^* = P(E)$ ,  $E \in \mathcal{B}(X)$ .

As a result, we note that for every  $f, g \in \mathcal{K}$ ,  $\mu_{f,g}(\cdot) = \langle f | P(\cdot) g \rangle$  is a signed Borel measure on  $X$ . We shall use the abbreviation  $\mu_f(E) := \mu_{f,f}(E) = \|P(E)f\|^2$ ,  $f \in \mathcal{K}$ ,  $E \in \mathcal{B}(X)$ , and we note that  $\mu_f$  is positive. Moreover

$$|\mu_{f,g}(E)|^2 \leq \mu_f(E) \mu_g(E), \quad f, g \in \mathcal{K}, E \in \mathcal{B}(X). \quad (4.14)$$

Formula (v) above specializes to the recursive identity

$$\sum_{i=1}^N \mu_{S_i^* f} \circ \tau_i^{-1} = \mu_f. \quad (4.15)$$

Note that this is the covariance condition (1.5) from the introduction. Substituting  $S_i f$  for  $f$ , we get

$$\mu_f \circ \tau_i^{-1} = \mu_{S_i f}. \quad (4.16)$$

We are now ready to define the operator  $W: \mathcal{K} \rightarrow \mathcal{H}(X)$  which intertwines the given representation of  $\mathcal{O}_N$  on  $\mathcal{K}$  with the universal  $\mathcal{O}_N$ -representation acting on  $\mathcal{H}(X)$ :

$$W: \mathcal{K} \ni f \longmapsto \mu_f \longmapsto (1, \mu_f) \in \mathcal{H}(X).$$

It follows from the definitions that

$$\|f\|_{\mathcal{K}}^2 = \|(1, \mu_f)\|_{\mathcal{H}(X)}^2, \quad (4.17)$$

where the norm  $\|\cdot\|_{\mathcal{H}(X)}$  is defined in Section 2. Using the polarization identity

$$\langle f | g \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|i^k f + g\|^2, \quad (4.18)$$

we further note that  $W$  is an isometry from  $\mathcal{K}$  to  $\mathcal{H}(X)$ . Moreover, formula (4.16) yields

$$WS_i^\mathcal{K}f = (1, \mu_f \circ \tau_i^{-1}) = S_i Wf, \quad (4.19)$$

where  $S_i$  denotes the isometry in  $\mathcal{H}(X)$  defined in (2.4), i.e., the universal representation.  $\square$

**Example 4.3** To see that there are examples which are covered by Theorem 4.2 but not by the more restrictive setting in Section 3, we only need to specify an  $(S_i)$  in, say,  $\text{Rep}(\mathcal{O}_2, \mathcal{K})$  and some  $e_0 \in \mathcal{K}$  such that the measure  $\mu := \mu_{e_0}$  does not satisfy the conditions (3.12). Here is a simple one: let  $\mathcal{K} := L^2(\mathbb{T})$ , where  $\mathbb{T}$  is the one-torus  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$  equipped with Haar measure. Set

$$S_0 f(z) = f(z^2), \quad S_1 f(z) = z f(z^2). \quad (4.20)$$

Then it is immediate that

$$S_i^* S_j = \delta_{i,j} I \quad \text{and} \quad \sum S_i S_i^* = I, \quad (4.21)$$

so (4.20) defines an element in  $\text{Rep}(\mathcal{O}_2, L^2(\mathbb{T}))$ . We take the corresponding IFS to be the unit interval with the subdivision from Example 3.4; see also Fig. 2. Setting  $e_n(z) = z^n$ , we note that

$$S_0 e_n = e_{2n}, \quad S_1 e_n = e_{2n+1}, \quad n \in \mathbb{Z},$$

so the two isometries permute the vectors in an orthonormal basis for  $\mathcal{K} = L^2(\mathbb{T})$ . If  $f \in \mathcal{K}$ , the measures  $\mu_f(\cdot) = \|P(\cdot)f\|^2$  are Borel measures on  $[0, 1]$ . Taking  $f = e_0$ , one easily checks that  $\mu_{e_0} = \delta_0$ ,  $\mu_{e_0} \circ \tau_0^{-1} = \delta_0$ , and  $\mu_{e_0} \circ \tau_1^{-1} = \delta_{1/2}$  where  $\delta_0$  and  $\delta_{1/2}$  are the Dirac measures at  $x = 0$  and  $x = 1/2$ , respectively, and  $\tau_k(x) := \frac{x+k}{2}$ ,  $k = 0, 1$ . This makes it clear that (3.12) is not satisfied. However, Theorem 4.2 does apply to this example.

**Remark 4.4** One might wonder how big a part of the universal Hilbert space  $\mathcal{H}(X)$  is needed for realizing the representations of  $\mathcal{O}_N$ . The answer is that the most general vectors  $\varphi\sqrt{d\mu}$  are needed if we want to represent all the elements in  $\text{Rep}(\mathcal{O}_N, \mathcal{K})$ . To see this, note that if  $(S_i)$  is in  $\text{Rep}(\mathcal{O}_N, \mathcal{K})$  for some Hilbert space  $\mathcal{K}$ , then the projections  $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$ ,  $k \in \mathbb{N}$ ,  $i_1, \dots, i_k \in \mathbb{Z}_N$ , generate a commutative  $C^*$ -algebra  $\mathfrak{A}$  of operators on  $\mathcal{K}$ . We showed in [14] that the corresponding representations of  $\mathfrak{A}$  include all possible spectral types. At the same time, Nelson showed in [10] that the spectral representation, including the multiplicity function, for abelian  $C^*$ -algebras  $\mathfrak{A}$  may be realized concretely in the Hilbert space  $\mathcal{H}(X)$  where  $X$  is the compact Gelfand space of  $\mathfrak{A}$ . Hence all possible positive measures on  $X$  will be needed in our understanding of the representations of  $\mathcal{O}_N$ : the representations of  $\mathcal{O}_N$  may be realized acting on vectors  $\varphi\sqrt{d\mu} \in \mathcal{H}(X)$ , and all positive Borel measures on  $X$  are needed for this.

## 5 Conclusions

This paper studies a general class of iterated function systems (IFS). No contractivity assumptions are made, other than the existence of some compact attractor. The possibility of escape to infinity is considered.

We are concerned with the realization of point transformations in Hilbert space. Our present approach in fact is based directly on a certain Hilbert-space construction, and on the theory of representations of the Cuntz algebras  $\mathcal{O}_N$ ,  $N = 2, 3, \dots$ .

While the more traditional approaches to IFS's start with some equilibrium measure, ours doesn't. Rather, we construct a Hilbert space directly from a given IFS, and our construction uses instead families of measures. Starting with a fixed IFS,  $\mathcal{S}_N$  with  $N$  branches, we prove existence of an associated representation of  $\mathcal{O}_N$ , and we show that the representation is universal in a certain sense. Our framework includes as a special case representations of  $\mathcal{O}_2$  associated with quadrature mirror filters and wavelets, and similarly, subband filters with  $N$  frequency subbands.

We further prove a theorem about a direct correspondence between a given system  $\mathcal{S}_N$ , and an associated sub-representation of the universal representation of  $\mathcal{O}_N$ .

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Our work is motivated by the lectures and papers by Arveson [13] and Tsirelson [11] from the symposium "Advances in Quantum Dynamics" held at Mount Holyoke in 2002, sponsored jointly by the AMS, IMS, and the SIAM. The paper [14] further offers more results on the general theory of representations of the Cuntz algebras. We thank Ola Bratteli for discussions. These papers make other interesting applications of the Hilbert spaces  $\mathcal{H}(X)$ . They also lists yet other applications of the idea underlying the  $\mathcal{H}(X)$  construction, and the book [10] by Nelson gives a beautiful presentation of the Spectral Representation Theorem, also couched in terms of the same Hilbert spaces.

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